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The existence of solutions for p-Laplacian boundary value problems at resonance on the half-line

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Abstract
By using the extension of the continuation theorem of Ge and Ren and constructing suitable Banach spaces and operators, we investigate the existence of solutions for a p-Laplacian boundary value problem with integral boundary condition at resonance on the half-line.

MSC: 34B40

Keywords: continuation theorem; integral boundary condition; resonance; p-Laplacian; boundary value problem

1 Introduction

A boundary value problem is said to be at resonance one if the corresponding homogeneous boundary value problem has non-trivial solutions. Mawhin's continuation theorem [1] is an effective tool to investigate the boundary value problems at resonance with linear or semilinear differential operators (see [2–7] and the references cited therein). Boundary value problems with p-Laplacian have been widely studied owing to their importance in theory and application of mathematics and physics (see [8–13]). But the p-Laplacian boundary value problems at resonance cannot be solved by Mawhin's continuation theorem. In order to solve these problems, Ge and Ren extended Mawhin's continuation theorem and used it to study boundary value problems with p-Laplacian [11]. In their new theorem, two projectors (Definition 2.2) P and Q must be constructed. But it is difficult to give the projector Q in many boundary value problems with p-Laplacian. In [14], the author extended the theorem in [11] and studied the problem

$$\begin{cases} (\varphi_p(u''))'(t) = f(t, u(t), u'(t), u''(t)), \quad t \in [0,1], \\ u''(0) = 0, \qquad u'(0) = \int_0^1 g(t)u'(t)\,dt, \qquad u'(1) = \int_0^1 h(t)u'(t)\,dt \end{cases}$$

in finite interval, where Q is not a projector but satisfies suitable conditions, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\int_0^1 g(t)\,dt = 1$, $\int_0^1 h(t)\,dt = 1$.

Boundary value problems on the half-line arise in various applications such as in the study of the unsteady flow of gas through a semi-infinite porous medium, in analyzing the heat transfer in radial flow between circular disks, in the study of plasma physics, in an analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, *etc.* [15]. In [16], using the continuation theorem of Ge and Ren [11], the author investigated the existence of solutions for the problem

$$\begin{cases} (\varphi_p(u'))' + f(t,u,u') = 0, & 0 < t < +\infty, \\ u(0) = 0, & \varphi_p(u'(+\infty)) = \sum_{i=1}^{n} \alpha_i \varphi_p(u'(\xi_i)) \end{cases}$$

on the half-line, where Q is a projector, $\alpha_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} \alpha_i = 1$.

In this paper, we study the boundary value problem

$$\begin{cases} (\varphi_p(u'))'(t) = \psi(t) f(t,u(t),u'(t)), & t \in [0,+\infty), \\ u'(+\infty) = 0, & u(0) = \int_0^{+\infty} h(t)u(t)\,dt \end{cases} \tag{1.1}$$

in infinite interval, where Q is not a projector, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. To the best of our knowledge, this is the first paper to study the boundary value problems at resonance on the half-line where the operator Q is not a projector.

In this paper, we will always suppose that the following conditions hold.

(H_1) $\int_0^{+\infty} h(t)\,dt = 1$, $th(t) \in L^1[0,+\infty)$, $\psi(t) \in L^1[0,+\infty) \cap C[0,+\infty)$, $h(t) \geq 0$, $\psi(t) > 0$, $t \in [0,+\infty)$.

(H_2) $f(t,u,v)$ is continuous in $[0,\infty) \times \mathbb{R}^2$. For any $r > 0$, there exists a constant $M_r > 0$ such that if $\frac{|u|}{1+t} \leq r$, $|v| \leq r$, $t \in [0,+\infty)$ then $|f(t,u,v)| \leq M_r$, and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t,u_1,v_1) - f(t,u_2,v_2)| < \varepsilon$ for $t \in [0,+\infty)$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2$, satisfying $\frac{|u_1-u_2|}{1+t} < \delta$, $|v_1 - v_2| < \delta$ and $\frac{|u_i|}{1+t} \leq r$, $|v_i| \leq r$.

The paper is organized as follows. The first section provides a short overview of the problem. Section 2 recalls some preliminary facts. Section 3 contains the main result of the paper.

2 Preliminaries

Definition 2.1 ([11]) Let X and Z be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Z$, respectively. An operator $M : X \cap \operatorname{dom} M \to Z$ is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Z,

(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\operatorname{dom} M$ denotes the domain of the operator M.

In this paper, an operator $T : X \to Z$ is said to be bounded if $T(V) \subset Z$ is bounded for any bounded subset $V \subset X$.

Definition 2.2 P is a projector if $P : Y \to Y$ is linear and $P^2x = Px$, where Y is a vector space.

Let $X_1 = \operatorname{Ker} M$, $P : X \to X_1$ be a projector and X_2 be the complement space of X_1 in X with $X = X_1 \oplus X_2$. Let $\Omega \subset X$ be an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.3 ([14]) Suppose that $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be M-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Z_1 of Z satisfying $\dim Z_1 = \dim X_1$ and two operators Q and R such that the following conditions hold:

(a) $\operatorname{Ker} Q = \operatorname{Im} M$,
(b) $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$,
(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$,
(d) $M[P+R(\cdot,\lambda)] = (I-Q)N_\lambda$,

where $Q: Z \to Z_1$ is continuous, bounded with $Q(I-Q) = 0$, $QZ = Z_1$ and $R: \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact with $Pu + R(u,\lambda) \in \operatorname{dom} M$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$.

Theorem 2.1 ([14]) *Let X and Z be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Z$, respectively, and $\Omega \subset X$ be an open and bounded nonempty set. Suppose that*

$$M: X \cap \operatorname{dom} M \to Z$$

is a quasi-linear operator and that $N_\lambda: \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is M-quasi-compact. In addition, if the following conditions hold:

(C_1) $Mx \neq N_\lambda x$, $\forall x \in \partial\Omega \cap \operatorname{dom} M$, $\lambda \in (0,1)$,
(C_2) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$,

then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$, *where $N = N_1$, $J: \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$, deg is the Brouwer degree.*

Remark In the proof of Theorem 2.1 in [14], the continuity of the operator M is not needed. And $\operatorname{dom} M$ may not be a linear space. But the operators P, R satisfy $Pu + R(u,\lambda) \in \operatorname{dom} M$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$.

Lemma 2.1 ([17]) *Let $\varphi_p: \mathbb{R} \to \mathbb{R}$ be a function given by the formula $\varphi_p(s) = |s|^{p-2}s$, where $p > 1$. Then, for any $u, v \geq 0$, we have*
(1) $\varphi_p(u+v) \leq \varphi_p(u) + \varphi_p(v)$, $1 < p \leq 2$.
(2) $\varphi_p(u+v) \leq 2^{p-2}(\varphi_p(u) + \varphi_p(v))$, $p \geq 2$.

3 Main results

In the following, we will always assume that q satisfies $1/p + 1/q = 1$.

Let $X = \{u \in C^1[0,+\infty): u'(+\infty) = 0, \sup_{t\in[0,+\infty)} \frac{|u(t)|}{1+t} < +\infty\}$ be endowed with the following norm $\|u\|_X = \max\{\|u'\|_\infty, \|\frac{u(t)}{1+t}\|_\infty\}$, $Y = \{y \in C[0,+\infty): \sup_{t\in[0,+\infty)} |y(t)| < \infty\}$ be endowed with the following norm $\|y\|_Y = \sup_{t\in[0,+\infty)} |y(t)| := \|y\|_\infty$. Take $Z = \{\psi y: y \in Y\} \times \mathbb{R}$, with norm $\|(\psi y, c)\|_Z = \max\{\|y\|_\infty, |c|\}$. We know that $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces.

Define operator $T: Y \to \mathbb{R}$ by $Ty = c$, for $y \in Y$, where c satisfies

$$\int_0^{+\infty} h(t) \int_0^t \varphi_q\left(\int_s^{+\infty} \psi(r)\big(y(r) - c\big)\,dr\right) ds\,dt = 0. \tag{3.1}$$

The following lemma shows that the operator T is well defined.

Lemma 3.1 *For $y \in Y$, there is only one constant $c \in \mathbb{R}$ such that $Ty = c$ with $|c| \leq \|y\|_\infty$, $T: Y \to \mathbb{R}$ is continuous and $T(ky) = kT(y)$, $k \in \mathbb{R}$.*

Proof For $y \in Y$, let

$$F(c) = \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big(y(r) - c\big)\,dr \right) ds\,dt.$$

Obviously, F is continuous and strictly decreasing in $\mathbb{R}$. If y is a constant, the results hold, clearly. Assume y is not a constant. Take $a = \inf_{t\in[0,+\infty)} y(t)$, $b = \sup_{t\in[0,+\infty)} y(t)$. It is easy to see that $F(a) > 0$, $F(b) < 0$. So, there exists a unique constant $c \in (a,b)$ such that $F(c) = 0$, *i.e.*, there is only one constant $c \in \mathbb{R}$ such that $Ty = c$ with $|c| \le \|y\|_\infty$.

For $y_1, y_2 \in Y$, assume $Ty_1 = a$, $Ty_2 = b$. By $h(t) \ge 0$, $\psi(t) > 0$, $\int_0^{+\infty} h(t)\,dt = 1$ and φ_q being strictly increasing, we obtain that if $b - a > \sup_{t\in[0,+\infty)}(y_2(t) - y_1(t))$, then

$$\begin{aligned} 0 &= \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big(y_2(r) - b\big)\,dr \right) ds\,dt \\ &= \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big[\big(y_1(r) - a\big) + \big(y_2(r) - y_1(r) - (b-a)\big)\big]\,dr \right) ds\,dt \\ &< \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big(y_1(r) - a\big)\,dr \right) ds\,dt = 0, \end{aligned}$$

a contradiction. On the other hand, if $b - a < \inf_{t\in[0,+\infty)}(y_2(t) - y_1(t))$, then

$$\begin{aligned} 0 &= \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big(y_2(r) - b\big)\,dr \right) ds\,dt \\ &= \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big[\big(y_1(r) - a\big) + \big(y_2(r) - y_1(r) - (b-a)\big)\big]\,dr \right) ds\,dt \\ &> \int_0^{+\infty} h(t) \int_0^t \varphi_q \left(\int_s^{+\infty} \psi(r)\big(y_1(r) - a\big)\,dr \right) ds\,dt = 0, \end{aligned}$$

a contradiction, too. So, we have $\inf_{t\in[0,+\infty)}(y_2(t) - y_1(t)) \le b - a \le \sup_{t\in[0,+\infty)}(y_2(t) - y_1(t))$, *i.e.*, $|b - a| \le \|y_2 - y_1\|_\infty$. So, $T : Y \to \mathbb{R}$ is continuous. Obviously, $T(ky) = kT(y)$, $k \in \mathbb{R}$. The proof is completed. □

Define operators $M : X \cap \operatorname{dom} M \to Z$, $N_\lambda : X \to Z$ as follows:

$$Mu(t) = \left[\big(\varphi_p(u')\big)'(t), T\left(\frac{(\varphi_p(u'))'(t)}{\psi(t)} \right) \right], \qquad N_\lambda u(t) = \big[\lambda \psi(t) f\big(t, u(t), u'(t)\big), 0\big],$$

where $\operatorname{dom} M = \{u \in X | \frac{(\varphi_p(u'))'}{\psi(t)} \in Y\}$.

Definition 3.1 u is a solution of (1.1) if $u \in \operatorname{dom} M$ satisfies (1.1).

It is clear that $u \in \operatorname{dom} M$ is a solution of (1.1) if and only if it satisfies $Mu = Nu$, where $N = N_1$.

Lemma 3.2 *M is a quasi-linear operator.*

Proof It is easy to get that $\operatorname{Ker} M = \{c | c \in \mathbb{R}\} := X_1$.

For $u \in X \cap \operatorname{dom} M$, if $Mu = (\psi y, c)$, then c satisfies (3.1) with y. On the other hand, if $y \in Y$, $Ty = c$, take

$$u(t) = -\int_0^t \varphi_q\left(\int_s^{+\infty} \psi(r)y(r)\,dr\right)ds.$$

By a simple calculation, we get $u \in X \cap \operatorname{dom} M$ and $Mu = (\psi y, c)$. Thus

$$\operatorname{Im} M = \bigl\{(\psi y, Ty)|y \in Y\bigr\} = \bigl\{(\psi y, c)|y \in Y, c \text{ satisfying (3.1) with } y\bigr\}.$$

By the continuity of T, we get that $\operatorname{Im} M \subset Z$ is closed. So, M is quasi-linear. The proof is completed. □

Take a projector $P : X \to X_1$ and an operator $Q : Z \to Z_1$ as follows:

$$(Pu)(t) = u(0), \qquad Q(\psi y, c) = (0, c - Ty),$$

where $Z_1 = \{(0, c)|c \in \mathbb{R}\}$. Obviously, $QZ = Z_1$ and $\dim Z_1 = \dim X_1$.

Define an operator R as

$$R(u,\lambda)(t) = -\int_0^t \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)f\bigl(r, u(r), u'(r)\bigr)\,dr\right)ds, \quad (u,\lambda) \in X \times [0,1].$$

Lemma 3.3 ([6]) *$V \subset X$ is relatively compact if $\{\frac{u(t)}{1+t}|u \in V\}$ and $\{u'(t)|u \in V\}$ are both bounded, equicontinuous on any compact intervals of $[0,+\infty)$ and equiconvergent at infinity.*

Lemma 3.4 *$R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact, $Pu + R(u,\lambda) \in \operatorname{dom} M$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$, where $X_2 = \{u \in X : u(0) = 0\}$, $\Omega \subset X$ is an open bounded set.*

Proof Firstly, we prove that $R : \overline{\Omega} \times [0,1] \to X_2$ and $Pu + R(u,\lambda) \in \operatorname{dom} M$, $u \in \overline{\Omega}$, $\lambda \in [0,1]$.

Obviously, $R(u,\lambda)(t) \in C^1[0,+\infty)$, $R(u,\lambda)'(+\infty) = -\lim_{t\to+\infty} \varphi_q(\int_t^{+\infty} \lambda\psi(s)f(s,u(s), u'(s))\,ds) = 0$. By ($H_2$), we get $\frac{|R(u,\lambda)(t)|}{1+t} \le \varphi_q(\|\psi\|_1 M_{\|u\|_X}) < +\infty$, $u \in X$. Therefore, $R(u,\lambda) \in X$. It is clear that $R(u,\lambda)(0) = 0$. Thus $R(u,\lambda) \in X_2$. Clearly, $R(u,\lambda) + Pu \in X$. It follows from $(\varphi_p(R(u,\lambda)(t) + Pu(t))')' = \lambda\psi(t)f(t,u(t),u'(t))$ and (H_2) that $\frac{(\varphi_p(R(u,\lambda)(t)+Pu(t))')'}{\psi(t)} = \lambda f(t,u(t),u'(t)) \in Y$. So, $R(u,\lambda) + Pu \in \operatorname{dom} M$.

Secondly, we show that R is continuous.

Since Ω is bounded, there exists a constant $r > 0$ such that $\|u\|_X \le r$, $u \in \overline{\Omega}$. By (H_2), there exists a constant $M_r > 0$ such that $|f(t,u(t),u'(t))| \le M_r$, $u \in \overline{\Omega}$, $t \in [0,+\infty)$. So, we get

$$\left|\int_t^{+\infty} \psi(s)f\bigl(s,u(s),u'(s)\bigr)\,ds\right| \le \|\psi\|_1 M_r, \quad t \in [0,+\infty), u \in \overline{\Omega}.$$

By the uniform continuity of $\varphi_q(x)$ in $[-\|\psi\|_1 M_r, \max\{1, \|\psi\|_1 M_r\}]$, we obtain that for any $\varepsilon > 0$, there exists a constant $\delta_\varepsilon > 0$ such that

$$\bigl|\varphi_q(x_1) - \varphi_q(x_2)\bigr| < \varepsilon, \quad |x_1 - x_2| \le \delta_\varepsilon, x_1, x_2 \in \bigl[-\|\psi\|_1 M_r, \max\bigl\{1, \|\psi\|_1 M_r\bigr\}\bigr].$$

For $\alpha = \frac{\delta_\varepsilon}{\|\psi\|_1}$, by ($H_2$), there exists a constant $\delta_\alpha > 0$ such that if $u, v \in \overline{\Omega}$, $\|u - v\|_X < \delta_\alpha$, then $|f(t, u(t), u'(t)) - f(t, v(t), v'(t))| < \alpha$, $t \in [0, \infty)$. So, we have

$$\begin{aligned}
&\left|\int_t^{+\infty} \psi(s) f\bigl(s, u(s), u'(s)\bigr)\, ds - \int_t^{+\infty} \psi(s) f\bigl(s, v(s), v'(s)\bigr)\, ds\right| \\
&\quad \leq \int_t^{+\infty} \psi(s) \bigl|f\bigl(s, u(s), u'(s)\bigr) - f\bigl(s, v(s), v'(s)\bigr)\bigr|\, ds \leq \delta_\varepsilon, \quad \|u - v\|_X < \delta_\alpha.
\end{aligned}$$

Take $\delta = \min\{\delta_\varepsilon, \delta_\alpha\}$. For $u, v \in \overline{\Omega}$, $\lambda, \mu \in [0, 1]$, if $\|u - v\|_X < \delta$, $|\lambda - \mu| < \delta$, then

$$\begin{aligned}
&\bigl|R(u, \lambda)'(t) - R(v, \mu)'(t)\bigr| \\
&\quad = \left|\varphi_q\left(\int_t^{+\infty} \lambda\psi(s) f\bigl(s, u(s), u'(s)\bigr)\, ds\right) - \varphi_q\left(\int_t^{+\infty} \mu\psi(s) f\bigl(s, v(s), v'(s)\bigr)\, ds\right)\right| \\
&\quad = \left|\varphi_q(\lambda)\varphi_q\left(\int_t^{+\infty} \psi(s) f\bigl(s, u(s), u'(s)\bigr)\, ds\right) - \varphi_q(\mu)\varphi_q\left(\int_t^{+\infty} \psi(s) f\bigl(s, v(s), v'(s)\bigr)\, ds\right)\right| \\
&\quad \leq \bigl[1 + \varphi_q\bigl(\|\psi\|_1 M_r\bigr)\bigr]\varepsilon, \quad t \in [0, +\infty).
\end{aligned}$$

This, together with

$$\begin{aligned}
&\frac{|R(u, \lambda)(t) - R(v, \mu)(t)|}{1 + t} \\
&\quad = \frac{|\int_0^t \varphi_q(\int_s^{+\infty} \lambda\psi(r) f(r, u(r), u'(r))\, dr)\, ds - \int_0^t \varphi_q(\int_s^{+\infty} \mu\psi(r) f(r, v(r), v'(r))\, dr)\, ds|}{1 + t} \\
&\quad \leq \frac{\int_0^t |\varphi_q(\int_s^{+\infty} \lambda\psi(r) f(r, u(r), u'(r))\, dr) - \varphi_q(\int_s^{+\infty} \mu\psi(r) f(r, v(r), v'(r))\, dr)|\, ds}{1 + t} \\
&\quad = \frac{\int_0^t |R(u, \lambda)'(s) - R(v, \mu)'(s)|\, ds}{1 + t} \leq \bigl\|R(u, \lambda)' - R(v, \mu)'\bigr\|_\infty,
\end{aligned}$$

means that $R : \overline{\Omega} \times [0, 1] \to X_2 \cap \operatorname{dom} M$ is continuous.

We will prove that $R : \overline{\Omega} \times [0, 1] \to X_2 \cap \operatorname{dom} M$ is compact.

It is easy to get that $\{\frac{R(u,\lambda)(t)}{1+t} : u \in \overline{\Omega}, \lambda \in [0, 1]\}$ and $\{R(u, \lambda)'(t) : u \in \overline{\Omega}, \lambda \in [0, 1]\}$ are bounded.

For any $T > 0$, $t_1, t_2 \in [0, T]$, $t_1 > t_2$, $u \in \overline{\Omega}$, $\lambda \in [0, 1]$, we have

$$\begin{aligned}
&\left|\frac{R(u, \lambda)(t_1)}{1 + t_1} - \frac{R(u, \lambda)(t_2)}{1 + t_2}\right| \\
&\quad = \left|\frac{1}{1 + t_2}\int_0^{t_2} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r) f\bigl(r, u(r), u'(r)\bigr)\, dr\right) ds\right. \\
&\qquad \left. - \frac{1}{1 + t_1}\int_0^{t_1} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r) f\bigl(r, u(r), u'(r)\bigr)\, dr\right) ds\right| \\
&\quad \leq \left|\frac{1}{1 + t_2} - \frac{1}{1 + t_1}\right| \varphi_q\bigl(M_r\|\psi\|_1\bigr) T + \varphi_q\bigl(M_r\|\psi\|_1\bigr)|t_2 - t_1|.
\end{aligned}$$

Since t and $\frac{1}{1+t}$ are uniformly continuous on $[0,T]$, we get that $\{\frac{R(u,\lambda)(t)}{1+t}, u \in \overline{\Omega}, \lambda \in [0,1]\}$ is equicontinuous on $[0,T]$.

$$\begin{aligned} &\bigl|R(u,\lambda)'(t_1) - R(u,\lambda)'(t_2)\bigr| \\ &\quad = \left|\varphi_q\left(\int_{t_2}^{+\infty} \lambda\psi(r)f\bigl(r,u(r),u'(r)\bigr)\,dr\right) - \varphi_q\left(\int_{t_1}^{+\infty} \lambda\psi(r)f\bigl(r,u(r),u'(r)\bigr)\,dr\right)\right|. \end{aligned}$$

Take $G(t) = \int_t^{+\infty} \lambda\psi(r)f(r,u(r),u'(r))\,dr$. We have

$$\bigl|G(t)\bigr| \le M_r\|\psi\|_1, \qquad \bigl|G(t_2) - G(t_1)\bigr| \le M_r\int_{t_2}^{t_1} \psi(r)\,dr.$$

It follows from the absolute continuity of integral and the uniform continuity of $\varphi_q(t)$ in $[-M_r\|\psi\|_1, M_r\|\psi\|_1]$ that $\{R(u,\lambda)'(t), u \in \overline{\Omega}, \lambda \in [0,1]\}$ is equicontinuous on $[0,T]$.

For $u \in \overline{\Omega}$, since

$$\int_s^{+\infty} \lambda\psi(r)\bigl|f\bigl(r,u(r),u'(r)\bigr)\bigr|\,dr \le M_r\int_s^{+\infty} \psi(r)\,dr$$

and

$$\lim_{s\to+\infty}\int_s^{+\infty} \psi(r)\,dr = 0,$$

for any $\varepsilon > 0$, there exists a constant $T_1 > 0$ such that

$$\varphi_q\left(\int_s^{+\infty} \lambda\psi(r)\bigl|f\bigl(r,u(r),u'(r)\bigr)\bigr|\right)\,dr < \frac{\varepsilon}{4}, \quad s > T_1, u \in \overline{\Omega}, \lambda \in [0,1].$$

Obviously, there exists a constant $T > T_1$ such that, for any $t > T$,

$$\frac{1}{1+t}\varphi_q\bigl(M_r\|\psi\|_1\bigr)T_1 < \frac{\varepsilon}{4}.$$

Thus, for any $t_1, t_2 > T$, we have

$$\begin{aligned} &\left|\frac{R(u,\lambda)(t_1)}{1+t_1} - \frac{R(u,\lambda)(t_2)}{1+t_2}\right| \\ &\quad = \left|\frac{1}{1+t_2}\int_0^{t_2} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)f\bigl(r,u(r),u'(r)\bigr)\,dr\right)ds\right. \\ &\qquad \left. - \frac{1}{1+t_1}\int_0^{t_1} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)f\bigl(r,u(r),u'(r)\bigr)\,dr\right)ds\right| \\ &\quad \le \frac{1}{1+t_2}\int_0^{T_1} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)\bigl|f\bigl(r,u(r),u'(r)\bigr)\bigr|\,dr\right)ds \\ &\qquad + \frac{1}{1+t_2}\int_{T_1}^{t_2} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)\bigl|f\bigl(r,u(r),u'(r)\bigr)\bigr|\,dr\right)ds \\ &\qquad + \frac{1}{1+t_1}\int_0^{T_1} \varphi_q\left(\int_s^{+\infty} \lambda\psi(r)\bigl|f\bigl(r,u(r),u'(r)\bigr)\bigr|\,dr\right)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1+t_1}\int_{T_1}^{t_1}\varphi_q\left(\int_s^{+\infty}\lambda\psi(r)\left|f\left(r,u(r),u'(r)\right)\right|dr\right)ds\\
&\le\frac{1}{1+t_2}\varphi_q\left(M_r\|\psi\|_1\right)T_1+\frac{t_2-T_1}{1+t_2}\frac{\varepsilon}{4}+\frac{1}{1+t_1}\varphi_q\left(M_r\|\psi\|_1\right)T_1+\frac{t_1-T_1}{1+t_1}\frac{\varepsilon}{4}<\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|R(u,\lambda)'(t_1)-R(u,\lambda)'(t_2)\right|\\
&=\left|\varphi_q\left(\int_{t_2}^{+\infty}\lambda\psi(r)f\left(r,u(r),u'(r)\right)dr\right)-\varphi_q\left(\int_{t_1}^{+\infty}\lambda\psi(r)f\left(r,u(r),u'(r)\right)dr\right)\right|\\
&\le\varphi_q\left(\int_{t_2}^{+\infty}\lambda\psi(r)\left|f\left(r,u(r),u'(r)\right)\right|dr\right)+\varphi_q\left(\int_{t_1}^{+\infty}\lambda\psi(r)\left|f\left(r,u(r),u'(r)\right)\right|dr\right)\\
&<\varepsilon.
\end{aligned}
$$

By Lemma 3.3, we get that $\{R(u,\lambda)|u\in\overline{\Omega},\lambda\in[0,1]\}$ is relatively compact. The proof is completed. □

Lemma 3.5 *Assume that $\Omega\subset X$ is an open bounded set. Then N_λ is M-quasi-compact in $\overline{\Omega}$.*

Proof It is clear that $\operatorname{Im}P=\operatorname{Ker}M$, $\operatorname{Ker}Q=\operatorname{Im}M$ and $QN_\lambda x=\theta$, $\lambda\in(0,1)\Leftrightarrow QNx=\theta$, *i.e.*, Definition 2.3(a) and (b) are satisfied.

For $u\in\Sigma_\lambda=\{\omega\in\overline{\Omega}\cap\operatorname{dom}M:M\omega=N_\lambda\omega\}$, by ($H_1$) and ($H_2$), we get

$$
\begin{aligned}
R(u,\lambda)&=-\int_0^t\varphi_q\left(\int_s^{+\infty}\lambda\psi(r)f\left(r,u(r),u'(r)\right)dr\right)ds\\
&=-\int_0^t\varphi_q\left(\int_s^{+\infty}\left(\varphi_p\left(u'\right)\right)'(r)\,dr\right)ds\\
&=u(t)-u(0)=(I-P)u.
\end{aligned}
$$

Clearly, $R(\cdot,0)=0$. Thus, Definition 2.3(c) is satisfied. For $u\in\overline{\Omega}$, we have

$$
M\left[Pu+R(u,\lambda)\right](t)=\left[\lambda\psi(t)f\left(t,u(t),u'(t)\right),T\left(\lambda f\left(t,u(t),u'(t)\right)\right)\right]=(I-Q)N_\lambda u(t).
$$

So, Definition 2.3(d) is satisfied.

Considering (H_2),

$$
\left\|N_\lambda(u)-N_\lambda(v)\right\|_Z\le\sup_{t\in[0,\infty)}\left|f\left(t,u(t),u'(t)\right)-f\left(t,v(t),v'(t)\right)\right|,\quad u,v\in\overline{\Omega},
$$

and

$$
\left\|N_\lambda(u)\right\|_Z\le\sup_{t\in[0,\infty)}\left|f\left(t,u(t),u'(t)\right)\right|,\quad u\in\overline{\Omega},
$$

we can obtain that N_λ is continuous and bounded in $\overline{\Omega}$.

It follows from the continuity and boundedness of T that Q is continuous and bounded in Z. By a simple calculation, we can obtain that $Q(I-Q)(\psi y,c) = (0,0)$, $(\psi y,c) \in Z$.

These, together with Lemma 3.4, mean that N_λ is M-quasi-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 3.1 *Suppose that* (H_1), (H_2) *and the following conditions hold*:

(H_3) *There exist constants* $c_0 > 0$ *and* $l > 0$ *such that*

$$\int_0^{+\infty} h(t) \int_0^t \varphi_q\left(\int_s^{+\infty} \psi(r) f\bigl(r,u(r),u'(r)\bigr)\,dr\right) ds\,dt \neq 0, \quad \bigl|u(t)\bigr| > c_0, t \in [0,l], u \in X.$$

(H_4) *There exist nonnegative functions* $a(t)$, $b(t)$, $c(t)$ *with* $(1+t)^{p-1}a(t)\psi(t), b(t)\psi(t), c(t)\psi(t) \in L^1[0,+\infty)$ *such that*

$$\bigl|f(t,x,y)\bigr| \le a(t)\bigl|\varphi_p(x)\bigr| + b(t)\bigl|\varphi_p(y)\bigr| + c(t), \quad \text{a.e. } t \in [0,+\infty),$$

where $\|(1+t)^{p-1}a(t)\psi(t)\|_1 l_0^{p-1} + \|b(t)\psi(t)\|_1 < 1$, *if* $1 < p \le 2$; $2^{p-2}\|(1+t)^{p-1}a(t) \times \psi(t)\|_1 l_0^{p-1} + \|b(t)\psi(t)\|_1 < 1$, *if* $p \ge 2$, *where* $l_0 = \max\{1,l\}$.

(H_5) *There exists a constant* $d_0 > 0$ *such that if* $|d| > d_0$, *then one of the following inequalities holds*:

(1) $df(t,d,0) > 0$, $t \in [0,l)$;

(2) $df(t,d,0) < 0$, $t \in [0,l)$.

Then the boundary value problem (1.1) *has at least one solution.*

In order to prove Theorem 3.1, we show two lemmas.

Lemma 3.6 *Assume* (H_1)-(H_4) *hold. Then the set*

$$\Omega_1 = \bigl\{u \in \operatorname{dom} M \mid Mu = N_\lambda u, \lambda \in (0,1)\bigr\}$$

is bounded in X.

Proof For $u \in \Omega_1$, we have $QN_\lambda u = 0$, *i.e.*, $T(f(t,u(t),u'(t))) = 0$. By (H_3), there exists a constant $t_0 \in [0,l]$ such that $|u(t_0)| \le c_0$. Since $u(t) = u(t_0) + \int_{t_0}^t u'(s)\,ds$, then

$$\frac{|u(t)|}{1+t} \le \frac{c_0 + |t-t_0|\|u'\|_\infty}{1+t} \le c_0 + \max\{1,l\}\bigl\|u'\bigr\|_\infty = c_0 + l_0\bigl\|u'\bigr\|_\infty, \quad t \in [0,+\infty).$$

Thus

$$\left\|\frac{u}{1+t}\right\|_\infty \le c_0 + l_0\bigl\|u'\bigr\|_\infty. \tag{3.2}$$

It follows from $Mu = N_\lambda u$, (H_4) and (3.2) that

$$\begin{aligned}&\bigl|\varphi_p\bigl(u'(t)\bigr)\bigr| \\ &\quad \le \int_0^{+\infty} \psi(t)\bigl[a(t)\bigl|\varphi_p\bigl(u(t)\bigr)\bigr| + b(t)\bigl|\varphi_p\bigl(u'(t)\bigr)\bigr| + c(t)\bigr]\,dt\end{aligned}$$

$$\leq \left\|\psi(t)a(t)(1+t)^{p-1}\right\|_1\varphi_p\left(\left\|\frac{u}{1+t}\right\|_\infty\right)+\|\psi b\|_1\varphi_p\left(\left\|u'\right\|_\infty\right)+\|\psi c\|_1$$
$$\leq \left\|\psi(t)a(t)(1+t)^{p-1}\right\|_1\varphi_p\left(c_0+l_0\left\|u'\right\|_\infty\right)+\|\psi b\|_1\varphi_p\left(\left\|u'\right\|_\infty\right)+\|\psi c\|_1.$$

Whenever $1 < p \leq 2$, by Lemma 2.1, we get

$$\left\|u'\right\|_\infty \leq \varphi_q\left(\frac{\|\psi c\|_1+\|\psi(t)a(t)(1+t)^{p-1}\|_1\varphi_p(c_0)}{1-\|\psi(t)a(t)(1+t)^{p-1}\|_1 l_0^{p-1}-\|\psi b\|_1}\right).$$

Whenever $p > 2$, by Lemma 2.1, we get

$$\left\|u'\right\|_\infty \leq \varphi_q\left(\frac{\|\psi c\|_1+2^{p-2}\|\psi(t)a(t)(1+t)^{p-1}\|_1\varphi_p(c_0)}{1-2^{p-2}\|\psi(t)a(t)(1+t)^{p-1}\|_1 l_0^{p-1}-\|\psi b\|_1}\right).$$

These, together with (3.2), mean that Ω_1 is bounded in X. □

Lemma 3.7 *Assume* (H_1)-(H_3) *and* (H_5) *hold. Then*

$$\Omega_2 = \{u \in \operatorname{Ker} M \mid QNu = 0\}$$

is bounded in X, *where* $N = N_1$.

Proof For $u \in \Omega_2$, we have $u = a$, $a \in \mathbb{R}$ and $Q(Nu) = 0$, *i.e.*,

$$\int_0^{+\infty} h(t)\int_0^t \varphi_q\left(\int_s^{+\infty}\psi(r)f(r,a,0)\,dr\right)ds\,dt = 0.$$

By (H_5), we get that $|a| \leq d_0$. So, Ω_2 is bounded. The proof is completed. □

Proof of Theorem 3.1 Let $\Omega = \{u \in X \mid \|u\| < r\}$, where $r > d_0$ is large enough such that $\Omega \supset \overline{\Omega}_1 \cup \overline{\Omega}_2$.

By Lemmas 3.6 and 3.7, we know $Mu \neq N_\lambda u$, $u \in \operatorname{dom} M \cap \partial\Omega$ and $QNu \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$.

Let $H(u,\delta) = \rho\delta u + (1-\delta)JQNu$, $\delta \in [0,1]$, $u \in \operatorname{Ker} M \cap \overline{\Omega}$, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(0,a) = a$,

$$\rho = \begin{cases} -1 & \text{if } (\mathrm{H}_5)(1) \text{ holds}, \\ 1 & \text{if } (\mathrm{H}_5)(2) \text{ holds}. \end{cases}$$

Define a function

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

For $u \in \operatorname{Ker} M \cap \partial\Omega$, we have $u = a \neq 0$. Thus

$$H(u,\delta) = \rho\delta a + (1-\delta)\bigl(-T\bigl(f(t,a,0)\bigr)\bigr).$$

If $\delta = 1$, $H(u,1) = \rho a \neq 0$. If $\delta = 0$, by $QNu \neq 0$, we get $H(u,0) = JQN(a) \neq 0$. For $0 < \delta < 1$, we now prove that $H(u,\delta) \neq 0$. Otherwise, if $H(u,\delta) = 0$, then

$$T\bigl(f(t,a,0)\bigr) = \frac{\rho\delta}{1-\delta}a. \tag{3.3}$$

Since $\|u\|_X = |a| = r > d_0$, for $t \in [0,l)$, by (H_5) and Lemma 3.1, we have

$$\operatorname{sgn}\bigl(af(t,a,0)\bigr) = \operatorname{sgn}\bigl(T\bigl(af(t,a,0)\bigr)\bigr) = \operatorname{sgn}\bigl[a\bigl(T\bigl(f(t,a,0)\bigr)\bigr)\bigr] = \operatorname{sgn}\biggl(\frac{\rho\delta}{1-\delta}a^2\biggr) = \operatorname{sgn}(\rho),$$

a contradiction with the definition of ρ. So, $H(u,\delta) \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$, $\delta \in [0,1]$.

By the homotopy of degree, we get that

$$\begin{aligned}
&\deg\Bigl(JQN, \Omega \cap \operatorname{Ker} M, 0\Bigr) \\
&\quad= \deg\Bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} M, 0\Bigr) = \deg\Bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} M, 0\Bigr) \\
&\quad= \deg\Bigl(\rho I, \Omega \cap \operatorname{Ker} M, 0\Bigr) = \deg\Bigl(\pm I, \Omega \cap \operatorname{Ker} M, 0\Bigr) = \pm 1 \neq 0.
\end{aligned}$$

By Theorem 2.1, we can get that $Mu = Nu$ has at least one solution in $\overline{\Omega}$. The proof is completed. □

4 Example

Example 4.1 Let us consider the following boundary value problem at resonance:

$$\begin{cases} (\varphi_p(u'))'(t) = \psi(t)f(t,u(t),u'(t)), & t \in [0,+\infty), \\ u'(+\infty) = 0, \qquad u(0) = \int_0^{+\infty} h(t)u(t)\,dt, \end{cases} \tag{4.1}$$

where $p = \frac{4}{3}$, $h(t) = e^{-t}$,

$$f(t,u,v) = \begin{cases} 0, & t > l, \\ (l-t)(u^{\frac{1}{3}} + \sin^{\frac{1}{3}} v), & t \leq l, u, v \in \mathbb{R}, \end{cases}$$

$\psi(t) = \frac{1}{3}(1+t)^{-\frac{4}{3}}(1+l_0)^{-\frac{4}{3}}e^{-t}$, $l_0 = \max\{1,l\}$. Take $a(t) = b(t) = (l-t)$, $c(t) = 0$, $d_0 = c_0 = 27$. By a simple calculation, we can obtain $\|(1+t)^{p-1}a(t)\psi(t)\|_1 l_0^{p-1} + \|b(t)\psi(t)\|_1 < 1$. It is easy to get that conditions (H_1)-(H_5) are satisfied. It follows from Theorem 3.1 that problem (4.1) has at least one solution.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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